Math 206B Lecture 17 Notes

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1 Odds, Ends, and Things to Come

1.1 Recap episode

We have learned a few things so far:

1. Representation theory of S_n
2. Combinatorics of Young tableau and combinatorial algorithms
   (a) RSK algorithm
   (b) Hook length formula
   (c) Stanley's formula and the Hilman-Grassl algorithm.

We will learn about two more things:

1. $\mathrm{GL}_N(\mathbb{C})$ representation theory
2. Symmetric functions.

These are related to further topics we won't cover, namely Schubert calculus and enumerative algebraic geometry.

1.2 Combinatorial questions about partitions

Theorem 1.1 (MacMahon, c.1900)**.** *Let $\mathrm{PP}(\lambda, c)$ be the set of plane partitions of shape λ numbers $\leq c$. Then $PP(a^b, c)$ (rectangles with numbers at most c) satisfies*

$$|\operatorname{PP}(\lambda, c)| = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.$$

Here is a q-version.

Theorem 1.2.
$$\sum_{A\in \mathrm{PP}(a^b,c)} q^{|A|} = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{(i+j+k-1)_q}{(i+j+k-2)_q},$$
where $(m)_q = (1-q^m)/(1-q)$.

So when $q \to 1$, we get the original theorem.

The number of Lozenge tilings of an $a \times b \times c$ hexagon is the number of $\mathrm{PP}(a^b, c)$. View the tilings as stacked 3-dimensional cubes, a 3-dimensional partition of an $a \times b \times c$ box. This is a bit strange. For any graph G, the number of perfect matching in G is the determinant of some matrix. We can view a tiling as a matching in terms of the dual graph. How can this determinant have a product formula?

1.3 Related ideas in representation theory

Theorem 1.3 (Frobenius, c.1902). *Let* ξ^α *be the character corresponding to* M^α, *let* $\lambda = (\lambda_1, \dots, \lambda_\ell)$, *let* $\rho = (\ell, \dots, 1)$ *be a staircase partition, and for a permutation* ω, *let* $\omega\alpha = (\alpha_{\omega(1)}, \alpha_{\omega(2)}, \dots)$. *Then*
$$\chi^\lambda = \sum_{\omega \in S_\ell} \mathrm{sign}(\omega) \xi^{\lambda - \omega\rho + \rho}.$$
Here, the convention is $\xi^\alpha = 0$ *for all* $\alpha \notin \mathbb{N}^\ell$ *(if we get negative entries, discard the term).*

This formula looks a little like a determinant. There is a relationship between these formulas.

Theorem 1.4 (Kostka). *Let* $K_{\lambda,\mu} = \#\,\mathrm{SSYT}(\lambda, \mu)$. *Then*
$$\xi^\mu = \sum_{|\lambda|=n} K_{\lambda,\mu}\chi^\lambda.$$

This explains the triangular nature of everything because you cannot have a semistandard Young tableau with shape λ and weight μ if $\lambda \leq \mu$. Forbenius's theorem gives a form for the inverse of the matrix
$$(K_{\lambda,\mu})^{-1}_{|\lambda|=|\mu|=n}.$$
These values are called the **Kostka numbers**. Here is what we will learn:

Definition 1.1. The **Schur functions** are $s_\lambda = \sum_{A\in \mathrm{SSYT}(\lambda)} x_a^{m_1(A)} \cdots x_N^{m_N(A)}$, where $m_i(A)$ is the number of i in A.

Theorem 1.5. *The* s_λ *are symmetric functions* $(\mathbb{C}[x_1, \dots, x_N]^{S_N})$.

Definition 1.2. The **elementary symmetric polynomials** are $e_\alpha = e_{\alpha_1} e_{\alpha_2} \cdots$.

Theorem 1.6 (Jacobi-Trudy)**.** *Using the convention that $e_r = 0$ for all $r < 0$,*

$$s_\lambda = \det[\tilde{e}_{\lambda_i+j-i}]_{i,j=1,\ldots,\ell}.$$

It turns out that the Jacobi-Trudy theorem is saying the same thing as Frobenius's theorem. Also, the definition of Schur functions is equivalent to Kostka's theorem.[1] The idea is then that

$$|\operatorname{PP}(a^b, c)| = s_{(a,b)}(\underbrace{1, 1, \ldots, 1}_{c}).$$

Since the Schur functions have a determinant formula, we can see why there is a determinant formula for Lozenge tilings. To get the q-analogue, we can look at $s_{(a,b)}(1, q, q^2, \ldots, q^{c-1})$.

This is the general picture we will be going through in the next few weeks.

[1]This is in the same sense that whoever came up with the definition of variance understood a lot about scalar products. The definition is designed to be compatible with the nice structure.